Investment with Taxes

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ABSTRACT

A serious analysis of taxes makes optimization models of investments much more complicated and more interesting. Because the portfolio strategy and value function depend on the holdings in each security at each tax basis, it is difficult to solve these problems exactly. Even numerical solutions are difficult except for very small problems, given the size of the state space and the necessary path-dependence of the strategies. This paper formulates some investment problems with taxes and presents the numerical solution for some simple examples.

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Taxes are or should be one of the most important concerns of many investors. Although a rising proportion of investments seems to be made by tax-exempt or tax-deferred investors (such as university endowments or most pension funds), fully taxable investors still represent a significant proportion of the market. In the US, the exact tax treatment varies from asset to asset and from investor to investor. Generally, we need to consider both state and federal taxes on both gains and income from coupons or dividends, with relatively complicated rules for whether gains are realized on sale or with the price change, for offsetting gains and losses across categories and across time, and for special events such as mergers or share dividends. After-tax returns do differ significantly from pre-tax returns, and there is a lot that investors can do to influence the amount of taxes they pay, e.g., by exploiting differences in tax treatment of securities with similar risk and return. Indeed, a great deal of literature on taxes is devoted to identifying these schemes and their theoretical and empirical impact on security market equilibrium (for example, Miller (1977), Miller and Scholes (1978), Litzenberger and Ramaswamy (1980), Schaefer (1982), Constantinides (1983), Constantinides and Ingersoll (1984), Dybvig and Ross (1986), and Brown and Dybvig (1986)). The purpose of this paper is to make some progress in analyzing the optimal dynamic investment policy in the presence of taxes, with an emphasis on asset mix with optimal realization of capital gains and losses.

One important strategy for reducing tax payments can be summarized by the following simple rule of thumb: defer capital gains indefinitely and realize losses immediately. This rule of thumb, however, is in conflict with another very important goal of investment, namely finding the optimal balance of risk and return. For example, if the market has been bullish for a while, failure to realize capital gains may imply taking on more risk than is tolerable.

This paper investigates an investment problem in the presence of taxes and restrictions on short sales, highlighting the trade-off between taxes and diversification. Conceptually, the problem is very complicated because the optimal policy depends on the whole history of investment, and consequently the problem grows very quickly in the number of assets and the number of time periods. We do not solve the problem analytically. Instead, we propose and implement numerical algorithms which can solve the problem when the number of periods and the number of securities are both very small.

The optimal investment policy of the taxable investor in our model determines an optimal balance of tax savings, risk, and return. A better understanding of the single-agent decision problem is of direct importance for practitioners, and may eventually lead to a better understanding of equilibrium and pricing as well.

Section 2 sets up the investment problem in a model with taxes and gives the basic properties of optimal strategies for the model. Section 3 explains two different algorithms which are used in the paper to solve the investment problem numerically. Section 4 discusses numerical examples and section 5 concludes with suggestion for possible future research.

1 A Model of Investment with Taxes

We will consider an agent facing the following stylized investment problem:

Problem 1 *Given* $C_{0-} \geq 0$*, choose* $N_{s,t} \geq 0$*, for* $s = 0, 1, \ldots, T-1$*, and* $t = s, s+1, \ldots, T$*, to maximize* $E\,u(C_T)$ *subject to*

$$(1) \quad (\forall t) \qquad C_t = C_{t-} - N_{t,t}P_t + \sum_{s=0}^{t}(N_{s,t-1} - N_{s,t})(P_t - \tau(P_t - P_s)),$$

$$(2) \quad (\forall t) \qquad C_{t-} = C_{t-1}r,$$

and

$$(3) \quad (\forall s) \qquad N_{s,s} \geq N_{s,s+1} \geq \ldots N_{s,T}.$$

In this problem we assume an investor optimizes with portfolio decisions at points of time $t = 0, 1, \ldots, T-1$, with liquidation for consumption at T. Before trading at time 0, the investor has cash equal to C_{0-}. There is no intermediate consumption and the investor's objective is to maximize von Neumann-Morgenstern expected utility $E[u(C_T)]$ of terminal cash subject to an intertemporal budget constraint. Since our results are numerical, not

theoretical, we will assume a particular utility function in each optimization, which we will always take to be a concave power utility function, which exhibits constant relative risk aversion. More generally, the numerical algorithm should work well with any sufficiently smooth strictly concave and increasing utility function, with or without intermediate consumption.

The investor pays taxes at a constant rate τ. There are two assets, a short riskless bond and a risky stock. The bond pays interest at a constant rate. The before-tax return (equal to one plus the before-tax *rate* of return) of a bond over a time interval between consecutive points in time will be denoted by R. The notation r in the intertemporal budget constraint refers to the after-tax return $r = R - \tau(R-1)$. P_t refers to the price of a share of the stock at time t.

Each choice variable $N_{s,t} \geq 0$ represents the number of shares purchased at time s and still in the portfolio at time t. Shares purchased at different dates are not perfect substitutes because they have different tax bases. Hence, they generate different amounts of tax on sale and are in effect different goods. The budget constraint (1) accounts for the cost of purchases and the after-tax proceeds of sales, and the budget constraint (2) gives the after-tax return to cash. The investor optimally sells all the shares at time T and therefore the solution will have $N_{s,T} = 0$ for all $s = 0, 1, \ldots, T$. The constraint (3) says the shares are being sold off over time and any new purchases would be indexed by the later time of purchase. C_{t-} and C_t, respectively, refer to the investor's cash position at t before and after trading.

In the investment problem we have made a number of simplifying assumptions. For example, we have assumed that the stock pays no dividends. Including dividends would force the realization of some taxable income. Including *proportional* dividends would be simple, but including a realistic dividend process may be more difficult. Some care must be taken to obtain the correct tax treatment of dividends (e.g., individuals need to pay taxes on dividends when they are received, but corporations can deduct a large percentage of dividend income for tax purposes).

Including transaction costs in the model would be only a slight complication. It is important to represent the appropriate tax treatment for trans-

action costs. This may differ depending on whether the costs are implicit in the bid-ask spread or explicit in commissions.

We have assumed that capital losses are credited immediately. In practice there are complex rules of carry-forward and carry-back. Our assumption of immediate credit for capital losses is at least consistent with the situation of an investor who has already realized large capital gains from other sources (e.g., a corporation which realizes capital gains from sales of land, plant, or buildings on a regular basis in the course of business).

Under the current tax code, the "wash sale rule" disallows deduction of capital losses if the same security was purchased within a month. In fact, there are many close substitutes in security markets so investors can probably buy another security with similar risk and tax characteristic without violating the wash sale rule. We do not include any wash sale constraint in our examples.

We have assumed the investor is not allowed to sell short shares of the stock. Constantinides (1983) has shown that given costless short sales with symmetric tax treatment, an investor can adjust portfolio risk optimally without selling shares that have appreciated in value and therefore the portfolio management problem can be separated from the tax timing problem. Even if the IRS does not treat a short sale as a constructive sale, short sales are costly and therefore the separation of portfolio management and tax timing is not valid.[1] The exact importance of this point is an empirical issue. Dickson and Shoven (1994) present empirical evidence that selling other securities in a large portfolio may substitute for short sales without a significant loss of diversification, although it is not obvious how this applies over the whole life of a portfolio or with consumption withdrawals. In general, the investor has to balance the gains of adjusting risk in the portfolio by selling shares that have appreciated in value against losses of realizing capital gains earlier.

[1]Of course, costliness of short sales does not justify their prohibition in our examples. It is the case that some investors are precluded by regulation or oversight from selling short.

We want to mention two basic properties of optimal strategies for the investment problem described above.

Property 1: It is always optimal to sell all shares which have depreciated in value.

Property 2: It is not optimal to realize gains by selling shares and buy additional shares simultaneously. When selling shares, it is optimal to sell the shares with the lowest basis first.

A proof of Property 1 is given by Constantinides (1983). The main assumptions are the absence of transaction costs, constancy of the tax rate over time, and the assumption that the investor can make use of current losses. Even if investors cannot make use of current losses, it will not do any damage to follow this rule provided losses can be carried forward without restriction, there are no transaction costs, and the tax rate is constant over time. Property 2 stops short of saying that it is always optimal to defer gains. If there is a need to rebalance the portfolio by selling shares, it is optimal for the investor to sell shares and realize gains, but in that case it would never be optimal to buy shares again immediately to increase the portfolio's risk, because this strategy is dominated by buying and selling fewer shares to defer part of the tax liability.

The problem described above is very complicated because an optimal strategy at a specific point of time depends on the distribution of shares with different tax bases in the investor's portfolio which in turn depends on the whole past history of share prices. Due to the complexity of the problem we have not yet discovered a general theoretical solution. Rather we assume a binomial process for the stock price and solve the model numerically using two different algorithms (these algorithms will be explained in section 3). We will present numerical examples in the next section.

In the binomial model the current stock price is the previous price times a factor that can take two values, u and d. Therefore, there are 2^T possible states. Analyzing the problem is difficult since the portfolio choice at a point in the stock price tree depends on the whole history of stock prices and

not just the stock price at the node. We will use the constant relative risk aversion utility function for the numerical examples, i.e.,

$$(4) \quad u(C_T) = \frac{C_T^{1-\gamma}}{1-\gamma}.$$

Next we discuss the numerical results we have obtained. Details of the optimization are contained in the subsequent section.

2 Numerical Examples

In this section we investigate the properties of optimal investment strategies using numerical examples. We will normalize both the investor's initial wealth and the price to be \$1.

We first look at a three period example. We vary the tax rate τ and risk aversion γ, and we take $u = 1.5$, $d = 0.9$, and $R = 1.1$. (This defines r implicitly as $R-\tau(R-1)$.) Figure 1 shows the binomial tree for this problem and enumerates the nodes. Table 1 gives the optimal number of shares at each node for four different tax rates $\tau = 0\%$, 10%, 20%, and 30% for each of three different coefficients of relative risk aversion $\gamma = 2, 3$, and 5. The number of shares held at each node is monotonely increasing in the tax rate. This is for two reasons. As the tax rate increases, first, equity's tax timing option becomes more valuable, and second, equity's one-period risk premium declines less than proportionally to the decline in variance.

It is interesting to note that when the share price has increased, the investor generally sells some shares. Absent taxes, an agent with constant relative risk aversion would want to hold a fixed proportion of wealth in the risky asset. An increase in the share price increases the risky asset proportion, leading to a sale. The same is true in most of our examples. The only exception to this rule in the example is the case where $\gamma = 2$ and $\tau = 0.3$, i.e., the risk aversion coefficient is small and the tax rate is high. In

this case, it would be desireable to sell some shares if there were no taxes, but the value of deferring the gain makes it optimal to stand pat. Selling shares when prices rise is inconsistent with the argument of Constantinides (1983), who has shown that it is never optimal to sell shares at a gain (until necessary for consumption) in a perfect capital market in which a short sale is not a constructive sale for tax purposes. However, in the presence of short selling restrictions, it may be optimal to sell some shares to reduce the portfolio's risk as this numerical example shows. When the share price has declined, the number of shares held in the current period is greater than or equal to the number of shares held in the previous period, consistent with the intuition that after a decline in the share price the optimal balance of the risky asset and the riskless asset implied by a constant relative risk aversion utility function requires in general additional purchase of shares of the risky asset.

Table 1 also gives the certainty equivalent value of the expected utility. The certainty equivalent value is the value of final wealth $C_{C,T}$ whose utility value $U(C_{C,T})$ is equal to the expected utility of the investor. The certainty equivalent value declines as the tax rate increases, because the increase in the tax rate deteriorates the investment opportunity faced by the investor. The reduction in the certainty equivalent value due to the tax rate of 10% is equal to 2.48% if $\gamma = 2$, 2.55% if $\gamma = 3$, and 2.68% if $\gamma = 5$. The same reduction due to the tax rate of 20% (resp. 30%) is equal to 4.89% (7.3%) if $\gamma = 2$, 5.11% (7.59%) if $\gamma = 3$, and 5.22% (7.76%) if $\gamma = 5$. The loss in certainty equivalent as taxes increase is roughly independent of the level of risk aversion. This is due to two offsetting effects. At large levels of risk aversion, the value of participating in equities in the first place was smaller, but also the value of the tax timing option is less.

In the presence of taxes, there is no unique notion of wealth at a point in time because the gross value of shares in the portfolio includes the value of taxes to be paid when they are sold. But in order to put the numerical examples into perspective we also calculated the ratio of the paper value of share holdings to the paper value of total wealth which is sum of the paper value of share holdings and cash on hand. These ratios are given in Table 2. The case of no tax conforms to the usual rule for power utility and i.i.d. returns: the ratio of risky asset holdings to total wealth is constant. This

intuition is preserved even in the presence of taxes in the sense that whenever the investor wants to increase the number of shares (at nodes 1, 3, and 7) the ratio of paper value of share holdings to the paper value of total wealth is fairly constant; the ratio is an increasing function of the tax rate for the same reason given when explaining Table 1. However, after an increase in the share price this ratio becomes higher than it is in the previous period. This is because deferral of gains makes it worth bearing more risk than is optimal in the absence of taxes.

In the second example, we look at the case where the expected after-tax return on the stock for one period is equal to the risk-free rate, so the attraction of stock to the investor comes solely from the existence of a tax timing option. The parameter values for this example are $r = 1.1$, and $\gamma = 5$. Table 3 gives the number of shares held at each node for this example. When there is no tax, it is not optimal to invest in the risky stock because it is dominated by the risk-free asset in the sense of second-order stochastic dominance (Hadar and Russell (1969)). With taxes, the option to pay taxes later makes the expected return on the stock greater than the risk-free rate if the stock is held for two period or longer. The higher the tax rate, the higher the value of this option is and this fact is illustrated by our numerical example. In the example shown in Table 3, the investor buys a certain number of shares in period 0 and holds them until the final period if the share price increases between period 0 and period 1, and sells all the shares and buys a *smaller* number of shares in period 0 and holds them until the final period if the share price decreases between period 0 and period 1 and the share price increases between period 1 and period 2. At node 7, there is only one more period until the final period, therefore there is no tax timing option, so the investment in the risky is always zero. The numbers of shares bought at nodes 1 and 3 are monotone increasing function of the tax rate illustrating the fact that the value of tax timing option is increasing as the tax rate increases.

Table 3 also reports the certainty equivalent values. The tax-timing option increases the certainty equivalent value by 0.01% when $\tau = 25\%$, and by 0.08% when $\tau = 50\%$. We conducted the same numerical experiment for a 4-period model, even though the results are not reported in the paper. For a 4-period model the increase in the certainty equivalent value is higher than

it is for a 3-period model; the increase is 0.027% when $\tau = 25\%$ and 0.18% when $\tau = 50\%$. These results show that the tax-timing option has a small value if the investment horizon is short.

3 Numerical Algorithms

In this section we explain the two numerical algorithms which are used to solve the investment problem.

The first algorithm is based on activities of the investor, that is, it first enumerates all the possible activities available for the investor and then solves the optimization problem. An activity is the combination of the purchase of a share and the subsequent sale at a stopping time. The sale is at a stopping time because it is not known at time of purchase when the shares will be sold, but we must know enough at any time to decide whether to sell (stop), which is the defining property of a stopping time. The number of all possible activities in which a share is purchased in period i is equal to the number of stopping times for the period $i+1$ through N. It is easy to show that the number $\mathcal{S}(n)$ of stopping times with n periods remaining satisfies the following recursive equation.

$$\mathcal{S}(n+1) \;=\; (1+\mathcal{S}(n))^2 \tag{5}$$

$$\mathcal{S}(1) \;=\; 1. \tag{6}$$

And the number of all possible activities for the N-period problem is given by

$$\sum_{k=1}^{N} 2^{N-k}\mathcal{S}(k), \tag{7}$$

which becomes a very large number very quickly; the first twenty values of the sum in (7) are approximately[2] 1, 6, 37, 750, 459829, $2.1e11$, $4.4e22$, $1.9e45$, $3.8e90$, $1.4e181$, $2.1e362$, $4.3e724$, $1.8e1449$, $3.3e2898$, $1.1e5797$, $1.2e11594$, $1.5e23188$, $2.4e46376$, $5.6e92752$, and $3.1e185505$. The values explode even more quickly if we have more risky assets or if the asset follows a trinomial or multinomial process. We are able to remove some activities by relying on properties 1 and 2 of the previous section which an optimal strategy must satisfy. With this constraint, we find that the numerical algorithm works nicely for $T \leq 4$ periods with a single binomial risky asset with our current computing capacity, and it is clear that even orders of magnitude of increase in computing power will not allow us to handle much larger problems. Given the list of all possible activities, the amount of final wealth is given by

$$(8) \quad C_T \;=\; C_{0-}\left[r^T + \sum_{j=1}^{M} a(j)(\pi(j) - r^T)\right]$$

where M is the total number of activities that satisfy property 2 of section 2, $a(j) \geq 0$ is the amount spent on the j-th activity, and $\pi(j)$ is the gross after-tax profit activity j will generate at the final time if \$1 is invested in activity j in period 0 (therefore, the actual amount invested in the purchase period i will be $a(j)r^i$). For example, when there are only two periods there are four different activities that satisfy properties 1 and 2 in the previous section, namely, the first activity is the activity in which shares are bought in period 0 and sold in period 1, the second activity is the one in which shares are bought in period 1 if and only if the share price has increased between period 0 and period 1, the third activity is the one in which shares are bought in period 0 and they are sold in period 1 if and only if the share price has declined between period 0 and period 1, and the fourth activity is such that shares are purchased in period 1 if and only if the share price has declined between period 0 and period 1 (Property 2 implies that it is not optimal to use the first and second activities simultaneously). Table 4 gives the net profits ($= \pi(j) - r^2$) which these activities generate in all possible states for the two-period problem. Figure 2 shows the stock price movements and the names of states that appear in Table 4. Note that the tax rate τ

[2]We use the usual notation of floating-point computers; $2.1e11 \equiv 2.1 \times 10^{11}$.

has an influence on the investor's optimal decision because the net profits depend on it.

We now return to the general case. The first algorithm solves the optimization problem

$$(9) \quad \max E\,U(C_T) \;=\; \sum_{i=1}^{N} p(i)U\left(C_{0-}\left[r^T + \sum_{j=1}^{M} a(j)(\pi(j)(i) - r^T)\right]\right),$$

where N is the total number of states in the final period, $p(i)$ is the probability of the i-th state and $\pi(j)(i)$ is the gross after-tax profit which activity j will generate at state i. Our numerical solution uses a public-domain FORTRAN program *conmin* distributed by the Association for Computing Machinery (ACM).

The second algorithm is based on the number of shares which will be kept at each state in each period. If the number of shares to be kept is known for each state in each period, there is no relevant ambiguity which shares are to be sold or to be kept because of property 2 in the previous section. First realize all losses, then sell as necessary to get to the required holding starting with the highest basis (largest loss), or buy at market. The amount of wealth at the final time is uniquely determined as a function of the numbers of shares to be kept at each state in each period. Therefore, the optimization is over the space of the vectors which have 2^N components and whose i-th component denotes the number of shares to be kept at the j-th state of period k if $i = 2^k + j$ with $1 \leq j < 2^k$ (We will call the j-th state of period k *node* i if $i = 2^k + j$). Numerical optimization requires computation of derivatives $\partial C_T/\partial b_i$ $1 = 1, \ldots, 2^N$, where b_i is the number of shares to be held at node i. We will use the two-period case to illustrate this computation. When $b_2 < b_1$, we have:

$$(10) \quad \frac{\partial C_T}{\partial b_1} = \begin{cases} (u - \tau(u-1))r - r^2 & \text{states 1 and 2} \\ (d - \tau(u-1))r - r^2 & \text{states 3, 4} \end{cases}$$

$$(11)\quad \frac{\partial C_T}{\partial b_2} = \begin{cases} (u^2 - \tau(u^2-1)) - (u - \tau(u-1))r & \text{state 1} \\ (ud - \tau(ud-1)) - (u - \tau(u-1))r & \text{state 2} \\ 0 & \text{states 3, 4} \end{cases}$$

$$(12)\quad \frac{\partial C_T}{\partial b_3} = \begin{cases} 0 & \text{states 1 and 2} \\ (d - \tau(d-1))d - dr & \text{states 3, 4} \end{cases}$$

And when $b_1 < b_2$, we have:

$$(13)\quad \frac{\partial C_T}{\partial b_1} = \begin{cases} (u^2 - \tau(u^2-1)) - (u - \tau(u-1))u + (u-r)r & \text{state 1} \\ (ud - \tau(ud-1)) - (d - \tau(d-1))r + (u-r)r & \text{state 2} \\ 0 & \text{states 3, 4} \end{cases}$$

$$(14)\quad \frac{\partial C_T}{\partial b_2} = \begin{cases} (u - \tau(u-1))u - ur & \text{state 1} \\ (d - \tau(d-1))d - ur & \text{state 2} \\ 0 & \text{states 3, 4} \end{cases}$$

$$(15)\quad \frac{\partial C_T}{\partial b_3} = \begin{cases} 0 & \text{states 1 and 2} \\ (d - \tau(d-1))d - dr & \text{states 3, 4} \end{cases}$$

States in the above equations refer to states in the final period (see Figure 1). The above equations show that when calculating the derivative of C_T with respect to b_i, it is necessary to consider carefully the change in the number of shares to be sold or bought at the node or at the subsequent nodes and the change in the amount of cash, etc. For example, let us consider $\partial C_T/\partial b_2$ for the case $b_2 < b_1$: the first term for states 1 and 2 represents the direct change in net profit as the investment in the stock changes in period 1, and the second term for states 1 and 2 represents the effect of a change in the number of shares to be sold in period 1 as the result of a change in b_2. The algorithm calculates the derivatives by keeping a careful track of cash, number of shares to be bought or sold at each node.

Technically, the second algorithm has a difficulty because the right-hand and left-hand derivatives of final wealth C_T with respect to b_j are different if

the number b_j is such that a slight decrease of it requires a sale of shares with a different basis than the shares which are sold when the number to be kept at the node (a node is a state in a period) is equal to b_j because of a difference in tax payment per sale of a share. The situation typically arises when the numbers of shares to be kept in two consecutive periods are very close. In fact, the above equations show that the right-hand and left-hand derivatives are different if $b_1 = b_2$ for the two period problem. Conceivably this difficulty can be overcome by approximating the kink by a smooth function.

As they should, the two algorithms give almost identical solutions and this tends to confirm the integrity of both algorithms. As noted above, the second algorithm has a convergence problem when the first algorithm gives a solution in which the numbers of shares to be kept in two consecutive periods are very close. Otherwise, the two algorithms work very nicely for 2–, 3–, and 4–period problems and give the same solution to several decimal places.

It is difficult to incorporate general features such as loss carry-forward, dividends, and transaction costs in the first algorithm, but it is fairly easy to do so in the second algorithm. Once the technical convergence problem is overcome by using numerical methods for maximizing concave functions that are not differentiable, it is possible to deal with these general features of tax codes. Of course, there remains the curse of dimensionality and for now we can solve only unrealistically small problems.

4 Conclusion

In this paper we analyzed an investment problem with taxes. The presence of taxes makes investors' optimal decisions complicated because an optimal balance between the motive to reduce tax payments and the motive to have the right risk exposure has to be incorporated in these decisions. Using numerical examples the paper shows how this balance is different for different tax rates. It also shows that the tax timing option itself can be valuable and induce investors to hold risky securities even if the after tax one period return is equal to the risk-free rate.

The investment problem with taxes considered in this paper is very complicated because the optimal strategy at a point in time depends on the whole history of security prices and trading activities. This paper makes a contribution by posing this complicated problem and presenting two numerical algorithms that can be used to solve the problem if the dimension of the problem is small enough.

At this point, the analysis does not provide an algorithm for the practitioner, although the small examples may suggest rules of thumb. For now, the practitioner is probably best served by heuristic methods such as optimization within a small space of sensible rules of thumb.

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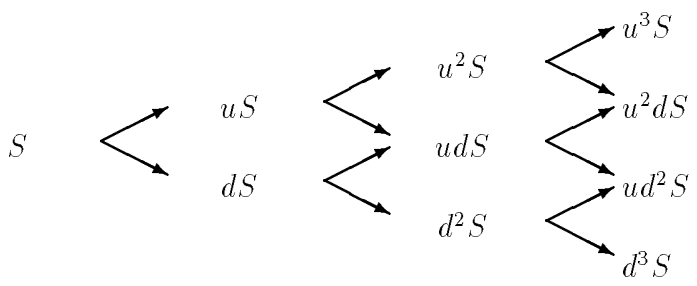
Figure 1

Stock Prices For A Three Period Problem



Node Names

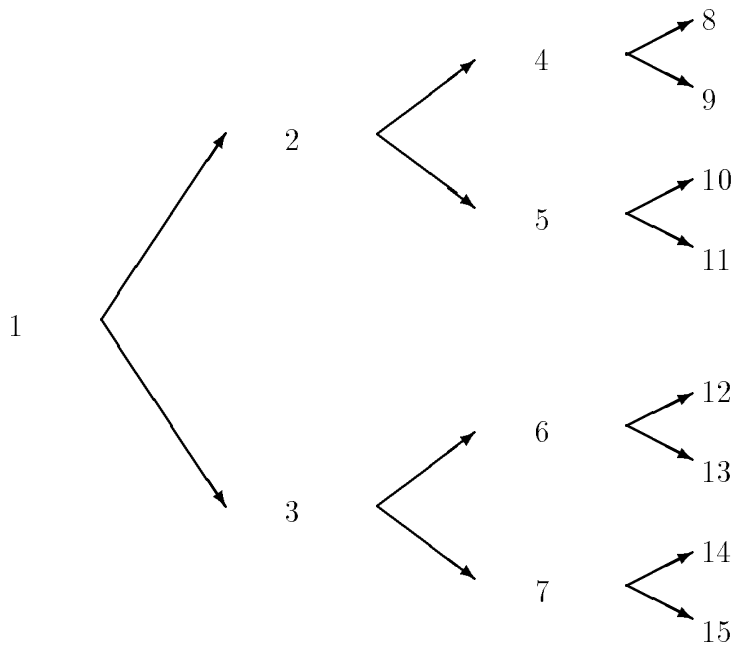


Table 1: The Number of Shares Held and The Certainty Equivalent Value At Each Node

	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7	CEV
$\gamma = 2$, $\tau = 0.0$	0.667	0.608	0.717	0.554	0.653	0.653	0.770	1.452
$\gamma = 2$, $\tau = 0.1$	0.731	0.700	0.780	0.646	0.703	0.734	0.833	1.416
$\gamma = 2$, $\tau = 0.2$	0.810	0.796	0.860	0.759	0.796	0.835	0.912	1.381
$\gamma = 2$, $\tau = 0.3$	0.915	0.915	0.962	0.902	0.915	0.962	1.01	1.346
$\gamma = 3$, $\tau = 0.0$	0.439	0.373	0.493	0.317	0.419	0.419	0.555	1.410
$\gamma = 3$, $\tau = 0.1$	0.480	0.428	0.536	0.368	0.452	0.471	0.600	1.374
$\gamma = 3$, $\tau = 0.2$	0.530	0.495	0.591	0.431	0.495	0.535	0.657	1.338
$\gamma = 3$, $\tau = 0.3$	0.596	0.571	0.660	0.510	0.571	0.617	0.732	1.303
$\gamma = 5$, $\tau = 0.0$	0.260	0.208	0.302	0.167	0.243	0.243	0.352	1.378
$\gamma = 5$, $\tau = 0.1$	0.284	0.239	0.329	0.193	0.262	0.272	0.381	1.341
$\gamma = 5$, $\tau = 0.2$	0.314	0.276	0.361	0.225	0.286	0.309	0.417	1.306
$\gamma = 5$, $\tau = 0.3$	0.352	0.322	0.404	0.266	0.322	0.356	0.464	1.271

Parameter value are as follows: $u = 1.5$, $d = 0.9$, the before-tax risk-free return $(r - \tau)/(1 - \tau)$ is equal to 1.1. CEV refers to the certainty equivalent value.

Table 2: The Ratio Of The Paper Value Of Stock Holdings To The Paper Value Of Total Wealth

	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7
$\gamma = 2, \tau = 0.0$	0.667	0.667	0.668	0.667	0.667	0.667	0.668
$\gamma = 2, \tau = 0.1$	0.731	0.756	0.732	0.750	0.722	0.745	0.735
$\gamma = 2, \tau = 0.2$	0.810	0.842	0.814	0.844	0.816	0.835	0.818
$\gamma = 2, \tau = 0.3$	0.915	0.938	0.919	0.943	0.927	0.941	0.919
$\gamma = 3, \tau = 0.0$	0.439	0.438	0.438	0.438	0.438	0.438	0.439
$\gamma = 3, \tau = 0.1$	0.480	0.500	0.480	0.500	0.478	0.483	0.483
$\gamma = 3, \tau = 0.2$	0.530	0.572	0.534	0.571	0.526	0.559	0.538
$\gamma = 3, \tau = 0.3$	0.596	0.648	0.602	0.652	0.607	0.638	0.610
$\gamma = 5, \tau = 0.0$	0.260	0.259	0.259	0.260	0.260	0.260	0.260
$\gamma = 5, \tau = 0.1$	0.284	0.298	0.285	0.299	0.284	0.293	0.286
$\gamma = 5, \tau = 0.2$	0.314	0.343	0.316	0.345	0.314	0.335	0.319
$\gamma = 5, \tau = 0.3$	0.352	0.397	0.356	0.402	0.356	0.387	0.361

Parameter values are as follows: $u = 1.5$, $d = 0.9$, the before-tax risk-free return $(r - \tau)/(1 - \tau)$ is equal to 1.1.

Table 3: The Number of Shares Held and The Certainty Equivalent Value For The Case Where The One-Period Expected After-Tax Return On The Stock Is Equal To The Risk-Free Rate

	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7	CEV
$\tau = 0$	0	0	0	0	0	0	0	1.33100
$\tau = 0.25$	0.0210	0.0210	0.0165	0.0210	0.0210	0.0165	0	1.33116
$\tau = 0.5$	0.0488	0.0488	0.0422	0.0488	0.0488	0.0422	0	1.33209

Parameter values chosen such that the after-tax returns on the stock over one period are the same and the expected value of these after-tax returns is equal to the risk-free rate. For the first row $u = 1.3$, $d = 0.9$, $r = 1.1$, $\gamma = 5$, for the second row $u = 1.4$, $d = 0.867$, $r = 1.1$, $\gamma = 5$, and for the third row $u = 1.6$, $d = 0.8$, $r = 1.1$, $\gamma = 5$.

Figure 2

Stock Prices And States For A Two-Period Problem

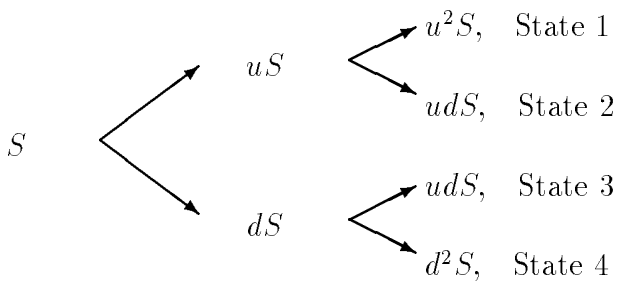


Table 4: Net Profits For A Two-Period Problem

	State 1	State 2	State 3	State 4
Activity 1	$(u-r+\tau(u-1))r$	$(u-r+\tau(u-1))r$	$(d-r+\tau(d-1))r$	$(d-r+\tau(d-1))r$
Activity 2	$(u-r+\tau(u-1))r$	$(d-r+\tau(d-1))r$	0	0
Activity 3	$u^2-r^2-\tau(u^2-1)$	$ud-r^2-\tau(ud-1)$	$(d-r+\tau(d-1))r$	$(d-r+\tau(d-1))r$
Activity 4	0	0	$(u-r+\tau(u-1))r$	$(d-r+\tau(d-1))r$