FIN 550J Exam Answers, October 21, 2010

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1. PROBABILITIES Let the stock price S take on values uniformly distributed on $[40, 60]$. A digital option on the stock has payoff

$$D = \begin{cases} 1 & \text{for } S > 55 \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

a. What is the density (pdf) of the stock price S?

$$f(S) = \begin{cases} 0 & \text{for } S < 40 \\ \frac{1}{60-40} = \frac{1}{20} & \text{for } 40 < S < 60 \\ 0 & \text{for } 60 < S \end{cases}$$

b. What is the distribution function (cdf) of the stock price S?

$$\begin{aligned} F(S) &= \int_{s=0}^{S} f(s)ds \\ &= \begin{cases} 0 & \text{for } S < 40 \\ \frac{S-40}{20} & \text{for } 40 \le S < 60 \\ 1 & \text{for } 60 \le S \end{cases} \end{aligned}$$

c. What is the distribution function (cdf) of the payoff D of the digital option?

$$\begin{aligned} G(D) &= \text{prob}(\tilde{D} \le D) \\ &= \begin{cases} 0 & \text{for } D < 0 \\ \text{prob}(S \le 55) = \frac{3}{4} & \text{for } 0 \le D < 1 \\ 1 & \text{for } 1 \le D \end{cases} \end{aligned}$$

d. Compute the mean, variance, and standard deviation of the digital option payoff.

$D = 0$ with probability 3/4 and 1 with probability 1/4. Therefore D has mean

$$E[D] = \frac{3}{4}0 + \frac{1}{4}1 = \frac{1}{4},$$

variance

$$\mathrm{var}[D] = E[D^2] - (E[D])^2 = \frac{3}{4}0 + \frac{1}{4}1 - \left(\frac{1}{4}\right)^2 = \frac{3}{16}$$

2. Linear equations (24 points) Consider the system of equations:

$$x_1 = 17 - 2x_2 - 3x_3$$

$$x_2 = 8 - 2x_1 - x_3$$

$$x_3 - 5 = x_1 - x_2$$

a. Write these equations in the form $Ax = b$. What are A and b?

Bringing all the expressions in x to the left-hand side and the constant terms to the right-hand side, we have

$$x_1 + 2x_2 + 3x_3 = 17$$

$$2x_1 + x_2 + x_3 = 8$$

$$-x_1 + x_2 + x_3 = 5$$

(This is not unique; you might have one or more of the equations multiplied by -1 on both sides, with the same changes to A and b below carried through.) Now, we can write this equation in matrix form as

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 17 \\ 8 \\ 5 \end{pmatrix}.$$

So, we can take

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 17 \\ 8 \\ 5 \end{pmatrix}.$$

b. Solve for x using Gaussian elimination.

We could do this by inverting A and then computing $x = A^{-1}b$, but it is less work to use b in the tableau instead. So, first set up the tableau

$$(A|b) = \begin{pmatrix} 1 & 2 & 3 & 17 \\ 2 & 1 & 1 & 8 \\ -1 & 1 & 1 & 5 \end{pmatrix}.$$

Subract twice the first row from the second row and add the first row to the third row to create zeros below the diagonal in the first column:

$$\begin{pmatrix} 1 & 2 & 3 & 17 \\ 0 & -3 & -5 & -26 \\ 0 & 3 & 4 & 22 \end{pmatrix}.$$

Add the second row to the third row to create another zero below the diagonal:

$$\begin{pmatrix} 1 & 2 & 3 & 17 \\ 0 & -3 & -5 & -26 \\ 0 & 0 & -1 & -4 \end{pmatrix}.$$

Add three times the third row to the first row and subtract five times the third row from the second row to create zeros above the diagonal in the third column:

$$\begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & -1 & -4 \end{pmatrix}.$$

Add 2/3 the second row to the first row to create another zero above the diagonal:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & -1 & -4 \end{pmatrix}.$$

Multiply the second row by $-1/3$ and the third row by -1 to convert the diagonal matrix to the identity:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{pmatrix}.$$

The solution is on the right;

$$x = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

3. OPTIMIZATION (26 points) Solve the following maximization problem:

Choose x_1 and x_2 to
maximize $10x_1 + 5x_2 - x_1^2 - x_1x_2 - x_2^2$
subject to $x_2 \leq 5$

Note: the second-order conditions are satisfied because the Hessian of theobjective function is negative definite and the constraint set is convex. You do not need to prove this.

First try the unconstrained maximization. Now,

$$\nabla f(x) = (10 - 2x_1 - x_2, 5 - 2x_2 - x_1).$$

Solving for $\nabla f(x) = 0$, we can write this as:

$$\begin{aligned} 10 - 2x_1 - x_2 &= 0 \\ 5 - x_1 - 2x_2 &= 0 \end{aligned}$$

Subtracting twice the second equation from the first, we have $3x_2 = 0$ so that $x_2 = 0$ and either equation then implies $x_1 = 5$. Since this unconstrained solution satisfies the constraint, it is also the constrained solution. So the solution is $x = (5, 0)^T$, and we are done.

If we tried the constrained maximization first, we would $g(x) = x_2$ and $\nabla g(x) = (0,1)$. The general Kuhn-Tucker conditions are

$$
\begin{aligned}
10 - 2x_1 - x_2 &= 0 \\
5 - x_1 - 2x_2 - \lambda &= 0 \\
\lambda(x_2 - 5) &= 0 \\
\lambda &\geq 0 \\
x_2 &\leq 5
\end{aligned}
$$

and if the constraint is binding in the solution then $x_2 = 5$ and we can solve for $x_1 = 5/2$ and $\lambda = -15/2$ but this contradicts $\lambda \geq 0$ so we can infer that there is no solution with the constraint binding. At this point, we would turn to the unconstrained problem.

4. EIGENVALUES AND EIGENVECTORS (30 points) Let

$$C = \begin{pmatrix} .5 & .4 \\ .5 & .6 \end{pmatrix}$$

A. Compute the eigenvalues of C.

The eigenvalues are the solutions of the characteristic equation $\det(A - \lambda I) = 0$. Since

$$A - \lambda I = \begin{pmatrix} .5 - \lambda & .4 \\ .5 & .6 - \lambda \end{pmatrix},$$

the characteristic equation is $(.5 - \lambda)(.6 - \lambda) - .4 \times .5 = 0$, or $\lambda^2 - 1.1\lambda + .1 = 0$. Factoring, we have $(\lambda - .1)(\lambda - 1) = 0$ or $\lambda = .1$ or 1.

Alternatively, factoring might be easier if we multiply by 10 first to get rid of the fractions: $10\lambda^2 - 11\lambda + 1 = 0$ or $(10\lambda - 1)(\lambda - 1) = 0$ and again $\lambda = .1$ or 1.

B. Compute the corresponding eigenvectors of C.

To compute an eigenvector, for each λ we find a solution of the equation

$$(A - \lambda I)x = 0.$$

For $\lambda = .1$, we have

$$\begin{pmatrix} .4 & .4 \\ .5 & .5 \end{pmatrix}.$$

The solution is only defined up to a multiple, so we try taking $x_2 = 1$ and we have $x_1 = -1$ so a corresponding eigenvector is $x = (-1, 1)^T$.

For $\lambda = 1$, we have

$$\begin{pmatrix} -.5 & .4 \\ .5 & -.4 \end{pmatrix}.$$

The solution is only defined up to a multiple, so we try taking $x_2 = 1$ and we find that $x_1 = .8$ so a corresponding eigenvector is $x = (.8, 1)^T$.

C. Compute $C^5(0, 1)^T$.

Now we can write

$$C^5\left(\alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} .8 \\ 1 \end{pmatrix}\right) = .1^5\alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 1^5\beta\begin{pmatrix} .8 \\ 1 \end{pmatrix}.$$

So, we are done if we can express $(0, 1)^T$ as a linear combination of the two eigenvectors. To do this, we solve for α and β:

$$\begin{pmatrix} -1 & .8 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which can be solved many ways for $(\alpha, \beta) = (4/9, 5/9)$. Therefore we have

$$\begin{aligned}
C^5\left(\alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} .8 \\ 1 \end{pmatrix}\right) &= .1^5\alpha\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 1^5\beta\begin{pmatrix} .8 \\ 1 \end{pmatrix} \\
&= .00001\frac{4}{9}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{5}{9}\begin{pmatrix} .8 \\ 1 \end{pmatrix} \\
&= .0000044444...\begin{pmatrix} -1 \\ 1 \end{pmatrix} + .55555...\begin{pmatrix} .8 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} .44444 \\ .55556 \end{pmatrix}
\end{aligned}$$

exactly.